Algebraic Properties of Elementary Cellular Automata

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Simple models – complex behavior

- One-dimensional maps (Ulam, 1950's)
- von Neumann's automata
- R. Coifman, GAFA2000 meeting:

> *... the state of science and technology around us is reaching a certain level of paralysis, in the sense, that complex phenomena are handled with very ad hoc methods, which are invented on the spot.*
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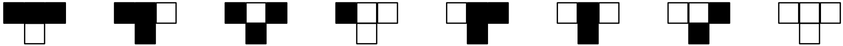

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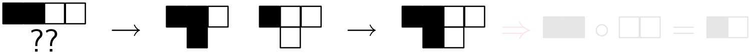

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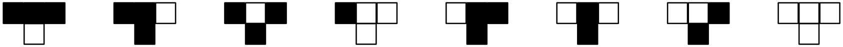

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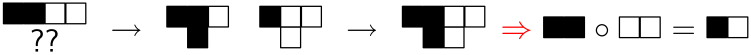

Denote $e_1 =$ ■■, $e_2 =$ ■□, $e_3 =$ □■, $e_4 =$ □□.
Products $e_i \circ e_j$, *consistent* with the corresponding rule define a *multiplication table*

$\circ$	e_1	e_2	e_3	e_4
e_1	e_4	e_3	e_1	e_2
e_2	e_1	e_1	e_3	e_4
e_3	e_2	e_1	e_1	e_2
e_4	e_1	e_1	e_3	e_4

or 4-element *groupoid* – a set of 4 elements together with a closed binary operation.

$$\text{ECA rule} \longrightarrow \text{groupoid}$$

A Central Problem

ECA evolution can be computed by groupoid multiplications of neighboring elements:

$$\begin{array}{ccccccc} x_1 & & & x_2 & & & x_3 \\ & x_1x_2 & & & & x_2x_3 & \\ & & & (x_1x_2)(x_2x_3) & & & \end{array}$$

Denote $B_{123} = (x_1x_2)(x_2x_3)$ ($\circ$ symbol is omitted).
Similarly, $B_{1234} = [(x_1x_2)(x_2x_3)][(x_2x_3)(x_3x_4)]$, which suggests a recursion

$$B_{12\dots n} = B_{12\dots(n-1)}B_{23\dots n}.$$

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In general, $n(n-1)/2$ products are required to calculate $B_{12\ldots n}$.

Can we do better than $O(n^2)$?

Can we do better than $O(n^2)$?

Yes (Pedersen, Moore) – if groupoid operation satisfy defining identities of known algebraic structures

- A *semigroup* is an *associative* groupoid, i.e. $x(yz) = (xy)z$
- A *group* is a semigroup with an *identity* element, and each element x has an *inverse* x^{-1}
- *Commutative* group: in addition $xy = yx$
- *Loops, quasigroups*

Example: rule 90

Multiplication table:

$\circ$	e_1	e_2	e_3	e_4
e_1	e_4	e_3	e_2	e_1
e_2	e_3	e_4	e_1	e_2
e_3	e_2	e_1	e_4	e_3
e_4	e_1	e_2	e_3	e_4

- associative
- e_4 is an identity element
- an inverse $x^{-1} = x$
- commutative
- $x^p = e_4$, if p is even,
  $= x$, if p is odd

$$
\begin{aligned}
B_{123} &= (x_1 x_2)(x_2 x_3) = x_1 x_2^2 x_3 = x_1 x_3 \\
B_{1234} &= [(x_1 x_2)(x_2 x_3)][(x_2 x_3)(x_3 x_4)] = x_1 x_2^3 x_3^3 x_4 = x_1 x_2 x_3 x_4,
\end{aligned}
$$

$$\boxed{B_{12\ldots n} = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n},} \qquad \text{where } p_k = \binom{k-1}{n-1},\ k = 1, 2, \ldots, n$$

However, a majority of ECA-groupoids do not belong to any known algebraic structures!

What kind of identities ECA-groupoids satisfy?

$$\Downarrow$$

Combinatorial search of groupoid identities

- *multi-homogeneous* identities, $m_1(x_1, x_2, \dots, x_k) = m_2(x_1, x_2, \dots, x_k)$,
  ex. $x_1(x_2x_3) = (x_1x_2)x_3$
- identities with B-blocks, $B_{12\dots k} = m(x_1, x_2, \dots, x_k)$,
  ex. $B_{122} = x_2(x_1x_2)$
- *trivial* identities $m(x_1, x_2, \dots, x_k) = e_j$,
  ex. $(x_1x_2)x_1 = e_2$

Multi-homogeneous Identities Search

$$m_1(x_1, x_2, ..., x_k) = m_2(x_1, x_2, ..., x_k),$$

m_1, m_2 are monomials of degree $d = d_1 + d_2 + \cdots + d_k$,
or products of x_i, each occurs d_i times, $i = 1, \ldots, k$.

Classify identities by partitions
$(d_1, d_2, \ldots, d_k)$, where $1 \leq d_1 \leq d_2 \cdots \leq d_k$.

Ex. $d = 3$, partition $(1, 2)$.

variables: x_1, x_2, x_2, and 3 permutations of them,

$$x_1x_2x_2,\ x_2x_1x_2,\ x_2x_2x_1.$$

2 *association types*: $(x_1x_2)x_2$ and $x_1(x_2x_2)$, thus 6 possible monomials:

$(x_1x_2)x_2$, $x_1(x_2x_2)$, $(x_2x_1)x_2$, $x_2(x_1x_2)$, $(x_2x_2)x_1$, $x_2(x_2x_1)$.

Search: evaluate monomials for all combinations of x_1 and x_2.

if $d = 8$, there are more than $17 \cdot 10^6$ monomials!

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Search: evaluate monomials for all combinations of x_1 *and* x_2.

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Identities: rules 4 and 8

Partition	Identities
	rule 4
(3)	$x_1x_1^2 = x_1^2x_1 = e_4$
(4)	$x_1(x_1x_1^2) \overset{(3)}{=} x_1(x_1^2x_1),\ \ (x_1x_1^2)x_1 \overset{(3)}{=} (x_1^2x_1)x_1$
(1, 3)	$B_{1222} = B_{2221},\ \ x_1(x_2x_2^2) \overset{(3)}{=} x_1(x_2^2x_2),\ \ (x_2x_2^2)x_1 \overset{(3)}{=} (x_2^2x_2)x_1$
(2, 2)	$B_{1122} = B_{2121},\ \ B_{1212} = B_{2211},\ \ x_1(x_1x_2^2) = x_2(x_2x_1^2),$ $(x_1^2x_2)x_2 = (x_2^2x_1)x_1$
(1, 1, 2)	$B_{1233} \overset{(1,1,1,1)}{=} B_{3231}$
(1, 1, 1, 1)	$B_{1234} = B_{4231}$
	rule 8
(3)	$x^2 = e_4,\ \ B_{111} = x_1x_1^2 = e_4$
(1, 2)	$B_{122} = B_{212} = B_{221} = x_1x_2^2 = x_2(x_1x_2) = e_4$
(1, 1, 1)	$B_{123} = B_{132} = B_{213} = B_{231} = B_{312} = B_{321},\ x_1(x_2x_3) = x_2(x_1x_3)$

Rule 30

Partition	Identities
(4)	$B_{1111} = x_1(x_1x_1^2) = x_1(x_1^2x_1)$
$(1,3)$	$B_{2221} = x_2(x_2^2x_1)$, $((x_2x_1)x_2)x_2 = (x_2^2x_2)x_1$
$(2,2)$	$((x_1x_2)x_2)x_1 = (x_1^2x_2)x_2$
$(1,1,2)$	$((x_1x_2)x_3)x_3 = ((x_1x_3)x_3)x_2$
(5)	$x_1(x_1(x_1x_1^2)) \overset{(4)}{=} x_1(x_1(x_1^2x_1))$, $(x_1(x_1x_1^2))x_1 \overset{(4)}{=} (x_1(x_1^2x_1))x_1$, $(x_1^2x_1)x_1^2 = ((x_1x_1^2)x_1)x_1$, $x_1^2(x_1^2x_1) = ((x_1^2x_1)x_1)x_1$
$(1,4)$	$x_1(x_2(x_2x_2^2)) \overset{(4)}{=} x_1(x_2(x_2^2x_2))$, $(x_2(x_2x_2^2))x_1 \overset{(4)}{=} (x_2(x_2^2x_2))x_1$, $x_2(((x_2x_1)x_2)x_2) \overset{(1,3)}{=} x_2((x_2^2x_2)x_1)$, $(((x_2x_1)x_2)x_2)x_2 \overset{(1,3)}{=} ((x_2^2x_2)x_1)x_2$, $x_2(x_1((x_2x_2^2)) = x_2(x_1(x_2^2x_2))$, $x_2(x_2((x_1x_2)x_2)) = x_2(x_2(x_2(x_2x_1)))$, $((x_1x_2)x_2)x_2^2 = ((x_1x_2^2)x_2)x_2$, $((x_2^2x_1)x_2)x_2 = ((x_2^2x_2)x_2)x_1$, $((x_2(x_1x_2))x_2)x_2 = (x_2^2x_2)(x_1x_2)$, $((x_2(x_2x_1))x_2)x_2 = (x_2^2x_2)(x_2x_1)$
$(2,3)$	18 identities

Rules 20, 54, and 110

Partition	Identities
	rule 20
(5)	$x_1(x_1(x_1x_1^2)) = x_1((x_1^2x_1)x_1),\ \ x_1^2(x_1x_1^2) = ((x_1^2x_1)x_1)x_1,$
	$(x_1(x_1x_1^2))x_1 = ((x_1x_1^2)x_1)x_1 = e_4$
(1, 4)	$(x_1x_2)(x_2(x_2x_2)) = (((x_1x_2)x_2)x_2)x_2,\ x_2((x_2(x_1x_2))x_2) = x_2((x_2(x_2x_1))x_2),$
	$((x_2(x_2x_1))x_2)x_2 = (x_2(x_2(x_2x_1)))x_2$
(2, 3)	$B_{22112} = B_{22121},\ (((x_1^2)x_2)x_2)x_2 = (((x_1x_2)x_2)x_1)x_2$
	rule 54
(4)	$B_{1111} = (x_1^2)^2 = e_4$
(5)	$(x_1(x_1^2x_1))x_1 = (x_1^2)^2x_1,\ x_1((x_1x_1^2)x_1) = x_1(x_1^2)^2$
(2, 3)	$(x_2x_1^2)x_2^2 = (x_2x_2^2)x_1^2,\ x_2^2(x_1^2x_2) = x_1^2(x_2^2x_2)$
(1, 2, 2)	$(x_1x_2^2)x_3^2 = (x_1x_3^2)x_2^2,\ x_3^2(x_2^2x_1) = x_2^2(x_3^2x_1)$
	rule 110
(4)	$B_{1111} = (x_1^2)^2 = e_4$
(5)	$(x_1(x_1^2x_1))x_1 = (x_1^2)^2x_1,\ x_1(x_1(x_1^2x_1)) = (x_1(x_1x_1^2))x_1$
(2, 3)	$(((x_1x_2)x_1)x_2)x_2 = (((x_2x_1)x_1)x_2)x_2$

Groupoid identities search: summary

- the lowest degree of identities found varies for different rules: rules 30, 45, 110 are of degree 4, rule 22 – of degree 6
- common identities: ECA groupoids of rules 30 and 45, also 54 and 110
- groupoids of class 1 and 2 ECA (rules 0, 8, 16, 24, 36) exhibit many identities with B-blocks
- groupoids of class 3 and 4 rules, such as 20, 30, 45, 52, 54, 73, 110 satisfy a small number of B-block identities, however ...

groupoid of rule 110 satisfies the following identities: of type $(1,6)$

$$B_{2212222} = (x_1x_2)(((x_2x_2^2)x_2)x_2),$$

of type $(1,7)$

$$B_{22122222} = B_{21222222} = B_{12222222} = (((x_1x_2)(x_2x_2^2))x_2)x_2^2 = (((x_1x_2)x_2^2)x_2^2)x_2^2 = (x_2(((x_1x_2)x_2^2)x_2))x_2^2 = ((x_1x_2)x_2^2)(x_2^2)^2 = x_2^2(((x_1x_2)x_2^2)x_2^2) = x_2^2(x_2(((x_1x_2)x_2^2)x_2)),$$

of type $(1,8)$

$$B_{222122222} = (x_1x_2)(((((x_2x_2^2)x_2)x_2)x_2)x_2) = (x_2((x_1x_2)x_2^2))(x_2^2)^2 = (x_1x_2)((x_2x_2^2)^2x_2) = (x_1x_2)(((x_2((x_2^2x_2)x_2))x_2)x_2) = (x_1x_2)((x_2x_2^2)(x_2(x_2^2x_2))) = (x_1x_2)(((x_2^2x_2)x_2^2)x_2^2) = (x_1x_2)((x_2(x_2^2)^2)x_2^2) = (x_1x_2)((x_2^2x_2)((x_2x_2^2)x_2)) = (x_1x_2)(x_2((x_2(x_2^2)^2)x_2)) = (x_1x_2)(x_2(((x_2^2x_2)x_2^2)x_2)) = (((x_2^2x_2)(x_2(x_1x_2)))x_2)x_2^2.$$

Beyond the empirical search:

Can we "derive" such higher-degree groupoid identities, starting from some finite set (a *basis*) of identities?

A *basis* is a set of identities from which all other groupoid identities can be derived.

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A *basis* is a set of identities from which all other groupoid identities can be derived.

4-element groupoids are not necessarily finitely based: some "true" identities cannot be derived from other (lower degree) identities, but can be "proved" by brute-force substitutions!

A Brief Review

2-element algebras are finitely based, R. Lyndon (1951).
However, in 1954 Lyndon discovered 7-element groupoid that is not finitely based.
Murskii (1965) found 3-element groupoid

$\circ$	e_1	e_2	e_3
e_1	e_1	e_1	e_1
e_2	e_1	e_1	e_2
e_3	e_1	e_3	e_3

with the following identity for any $n \geq 3$:

$$x_1(x_2(x_3 \ldots (x_{n-1}(x_n x_1) \ldots)) = (x_1 x_2)(x_n(x_{n-1} \ldots (x_4(x_3 x_2)) \ldots)),$$

that cannot be derived from any set of lower degree identities!

R. McKenzie (1997)

There is no recursive algorithm which when presented with an effective description of a finite groupoid will determine whether it is finitely based or not.

Other Structures

- all finite groups are finitely based,
  Oates and Powell (1965)
- all commutative semigroups are finitely based,
  Perkins (1968)
- However, the following 6-element semigroup

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where the operation is matrix multiplication, is not finitely based, Perkins (1968).

6 is the least order of a semigroup without a finite basis for identities, Trahtman (1983).

Compare:

regardless of the initial condition, a semigroup of at least 6 elements is required to obtain patterns more complicated than nested, NKS (p.887).

There are 10 ECA-semigroups that exhibit trivial behavior and are easily predictable.

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Summarizing,

- less “structure”, nice algebraic properties of groupoids implies more interesting ECA patterns, difficult to predict
- an important role of nonassociativity

Algebraic Cellular Automata (ACA)

- Besides 256 ECA, there might be other 4-element groupoids (4-groupoids) with interesting behavior – ACA. There are $4^{16} = 4,294,967,296$ possible 4-groupoids, and only 128 among them are distinct semigroups, G. Forsythe (1955).
- There are $3^9 = 19,683$ 3-groupoids, with 18 non-equivalent semigroups. Complicated patterns?
- 2-groupoids are finitely based – models of "simplicity"?

2- and 3-groupoids

Enumeration

- Use digits 0, 1, . . . instead of e_1, e_2 . . . .
- The following 3-element groupoid, for example,

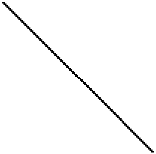
$\circ$	0	1	2
0	0	1	2
1	1	2	0
2	1	0	2

  will be numbered as a decimal 4061, since $12120102_3 = 4061_{10}$, and the number in base 3 is formed by rows of the multiplication table, starting from the top one.

Elements 2-groupoid: $0 = \square\square$, $1 = \blacksquare\blacksquare$
3-groupoid: $2 = \square\square$, $1 = {\color{blue}\blacksquare\blacksquare}$, $0 = {\color{red}\blacksquare\blacksquare}$

2-groupoids

3



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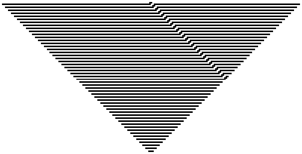

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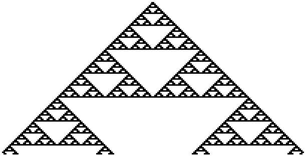

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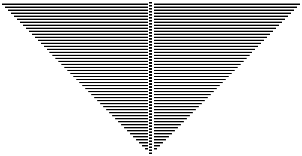
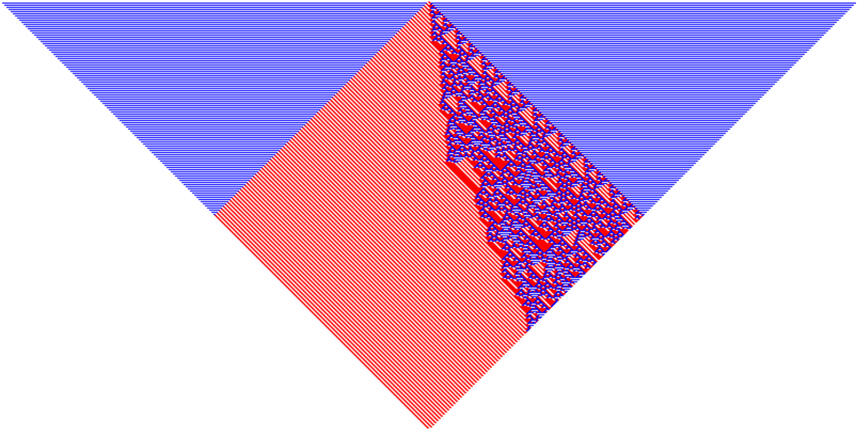
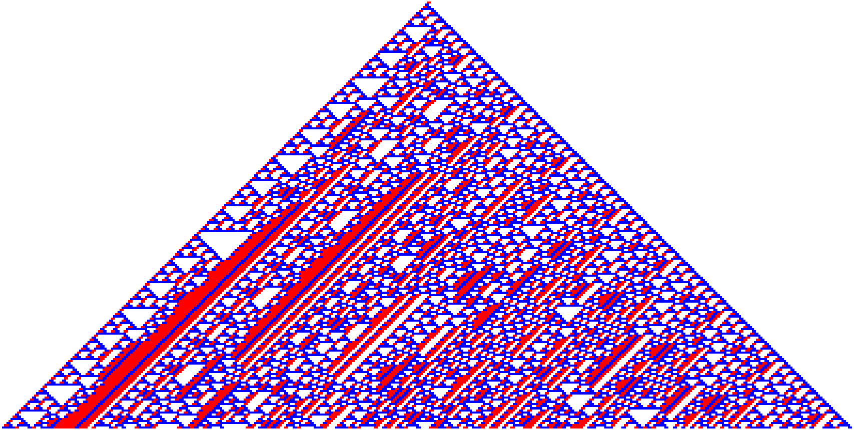


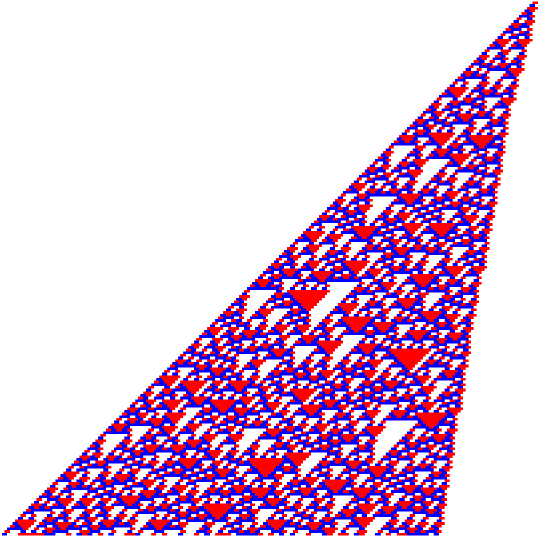
3-groupoid 2611



3-groupoid 3845



3-groupoid 4061



Conclusion

- Correlations between structural algebraic properties properties, patterns behavior, and predictability of ECA.
- “Symmetries” might not be adequate to explain complex phenomena.
- An important role of *experimental* mathematics: limitations of axiomatic approach.
- Nonassociative structures – what kind of science?

Jacobson's classical "Basic Algebra I" textbook states:

> *If we play ... axiomatic game with the concept of an associative algebra, we are likely to be led to the concept of a non-associative algebra, which is obtained simply by dropping the associative law of multiplication. If this stage is reached in isolation from other mathematical realities, it is quite certain that one would soon abandon the project, since there is very little of interest that can be said about non-associative algebras in general.*

Tarski's HSI Problem

$$
\begin{gathered}
a + b = b + a, \\
a + (b + c) = (a + b) + c, \\
a \cdot 1 = a, \quad a \cdot b = b \cdot a, \\
a \cdot (b \cdot c) = (a \cdot b) \cdot c, \\
a \cdot (b + c) = a \cdot b + a \cdot c
\end{gathered}
\qquad
\begin{gathered}
1^a = 1, \quad a^1 = a, \\
a^{b+c} = a^b \cdot a^c, \\
(a \cdot b)^c = a^c \cdot b^c, \\
(a^b)^c = a^{b \cdot c}
\end{gathered}
$$

Are there any laws in addition, multiplication add exponentiation that are true for natural numbers but do not follow from the familiar HSI?

Tarski's HSI Problem

$$\begin{gathered} a+b=b+a, \\ a+(b+c)=(a+b)+c, \\ a\cdot 1=a, \quad a\cdot b=b\cdot a, \\ a\cdot(b\cdot c)=(a\cdot b)\cdot c, \\ a\cdot(b+c)=a\cdot b+a\cdot c \end{gathered}$$

$$\begin{gathered} 1^a=1, \quad a^1=a, \\ a^{b+c}=a^b\cdot a^c, \\ (a\cdot b)^c=a^c\cdot b^c, \\ (a^b)^c=a^{b\cdot c} \end{gathered}$$

Are there any laws in addition, multiplication add exponentiation that are true for natural numbers but do not follow from the familiar HSI?

$$\begin{aligned} &((1+a)^a+(1+a+a^2)^a)^b\cdot((1+a^3)^b+(1+a^2+a^4)^b)^a = \\ &((1+a)^b+(1+a+a^2)^b)^a\cdot((1+a^3)^a+(1+a^2+a^4)^a)^b \end{aligned}$$

Wilkie (1981)