THE GOLDEN MEAN SHIFT IS THE SET OF $3x + 1$ ITINERARIES

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The iterated function system (IFS) based on the Collatz function is shown to be equivalent (for any given natural number) to two lattice paths determined by the number—one of which (called here the itinerary of the number) is closely related to the parity sequences defined by Everett [2]. If the two paths cross each other the number under consideration eventually reaches 1 through the repeated application of the Collatz IFS (as do all of the numbers that have been tested thus far by computer). If the lattice paths manage to avoid each other we have a counterexample to the Collatz conjecture (and none has been observed thus far). Of the two lattice paths one has the regularity of a straight line and this has been noticed at least by Wolfram [8]. The behaviour of the other path however, appears highly irregular at first and sensitive to its initial conditions (the number whose itinerary it is). It is shown here that (at least as a whole) the set of itineraries displays significant regularity. It forms the golden mean shift, a set well known in symbolic dynamics and described by Lind and Marcus [6], Huang and Scully [4] among others, with a self-similarity dimension of $-\log_2(\sqrt{5}-1)$ and an entropy equal to $\log_2 \frac{1+\sqrt{5}}{2}$. This result is new, but based on the work of Everett. Since we show that the itinerary of a number is directly related to (if not solely responsible for) the chaotic behavior of the Collatz IFS, one might want to investigate the relationship between a number and its itinerary. To that end we present a graph. The approach and the plotted picture presented are also both new. The graph presents a (striking) self-similarity and it sheds light on a question as complicated as the Collatz conjecture itself: what is the relationship between a positive integer and the number of times through the Collatz IFS before the number reaches 1 (regardless of whether the conjecture is true or false).

1 Introduction

1.1 The Function $f(2n) = n$, $f(2n+1) = 3n+2$

Under the iteration of the function

$$f(m) = \begin{cases} m/2 & m \text{ even} \\ (3m+1)/2 & m \text{ odd} \end{cases} \tag{1}$$

every integer $m \geq 0$ gives rise to an infinite sequence of integers.

$$m \to [m_0, m_1, m_2, \dots] \tag{2}$$

where $m_n = f^n(m)$. Thus for $m = 7$, one finds that

$$7 \to [7, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1, 2, \dots]$$

with $f^{11}(7) = 1$. It has been conjectured that *every* $m \geq 0$ has an iterate $f^k(m) = 1$. In 1977 Everett [2] stated two important facts about this function. First, that almost every integer m has an iterate $f^k(m) < m$. This is a density theorem whose object is the asymptotic behaviour of f when m approaches infinity. He also stated a second result (which was actually listed first in his paper) and that will be of interest here. To demonstrate it we will need additional terminology.

1.2 The Parity Sequence

The sequence (2) may be used to assign to every integer $m \geq 0$ a *parity sequence*

$$m \to \{x_0, x_1, x_2, \ldots\} \tag{3}$$

where $x_n = 0$ if $m_n = f^n(m)$ is even, and $x_n = 1$ if it is odd. For example, one sees from above that

$$7 \to \{1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, \ldots\}$$

The 2^N parity sequences for the integers $m < 2^N$ have subsequences $\{x_0, \ldots, x_{N-1}\}$ ranging over the full set of 2^N binary $(0, 1)$ vectors. Moreover, all integers $m = a + 2^N Q, Q = 0, 1, 2, \ldots$ have identical parity sequences through component x_{N-1}.

Theorem (Everett, 1977) *An arbitrary diadic sequence* $\{x_0, \ldots, x_{N-1}\}$ *arises via* (3) *from a unique integer* $m < 2^N$. *Specifically, the* x_n *determine* m *and* m_N *as follows*

$$m = a_{N-1} + 2^N Q_N \qquad 0 \leq a_{N-1} < 2^N \tag{4}$$

$$m_N = b_{N-1} + 3^X Q_N \qquad 0 \leq b_{N-1} < 3^X,\ X = \sum_{0}^{N-1} x_n \tag{5}$$

Hence the correspondence (3) *is one to one.*

Corollary *The correspondence*

$$m \to [m_0, \ldots, m_{N-1}] \to \{x_0, \ldots, x_{N-1}\}$$

induces a one to one mapping of all positive integers $m < 2^N$ *on the set of all* 2^N *diadic vectors* $\{x_0, \ldots, x_{N-1}\}$

2 Collatz Itineraries

2.1 The Collatz Function

Despite of all of the above, the original Collatz function was

$$f(x) = \begin{cases} \frac{x}{2} & x \text{ is even} \\ 3x + 1 & \text{otherwise} \end{cases} \tag{6}$$

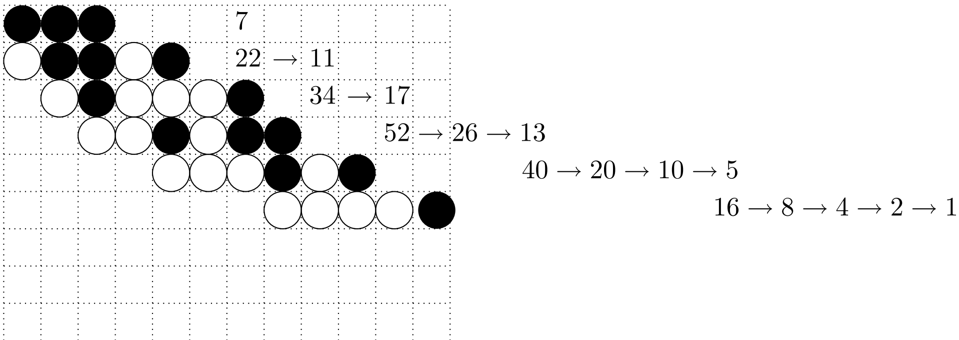
And it will be this function that we will concentrate on in the remaining of this paper. Wirsching [7] suggests using T (instead of f) to denote the function discussed by Everett (also by Terras, Lagarias, Berg and Meinardus) leaving f for (6).

2.2 Collatz Trajectories

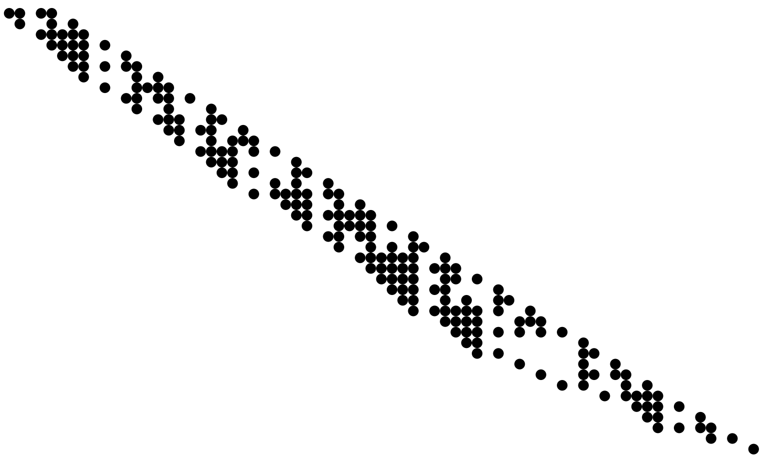
Definition (6) assigns to every integer $m \geq 0$ a *trajectory* in the Collatz IFS

$$m \to \{m = f^0(m), f^1(m), f^2(m), \ldots\} \tag{7}$$

in which we take $f^0(m)$ to be the same as m. We work out the following representation of a trajectory. Represent each number in base 2 then take the mirror image of the representation and work with it. Thus 4 (which is 100_2) will be represented as 001, 10 (which is 1010_2) will be shown as 0101, and so on. And starting from any number we calculate the effect of f on it: so if the number is even (and we need to divide by 2) we simply discard the leftmost digit (of necessity a zero, making the number even in the first place). If the leftmost digit is not zero the number is odd. In that case we multiply the number by 3, add one to it, and display the result on the following line, keeping the same representation (mirror image of the binary representation) for the result. Here's the trajectory of 7 under the Collatz iterator:



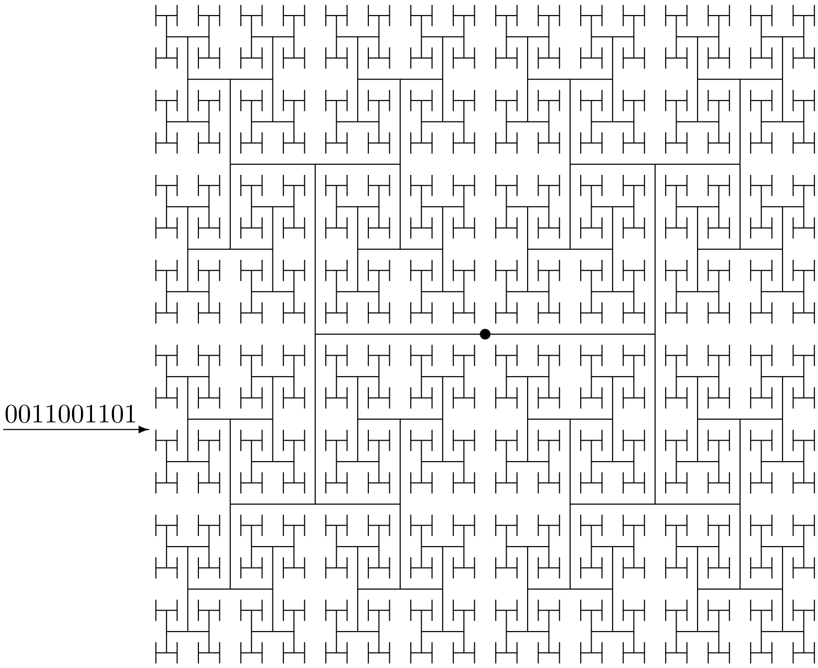
Here's the trajectory of 27 shown at a different scale and without annotations:



The right hand side (eastern coast) of this picture appears to be a straight line. And indeed, the bigger the original number, the closer we get to a straight line with an average slope of about $\log_3 2$. And that's because the multiplication by 3 becomes a lot more significant (compared to the addition of one) as the numbers get bigger.

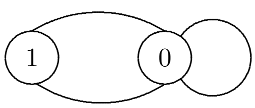
2.3 *Plotting Collatz Itineraries*

The approach we will take maps binary sequences onto the plane. Each binary sequence will be the address of a point in the following infinite structure (of which a finite part only, consisting of points with addresses up to 10 bits, is shown below):



The center of symmetry of this picture (shown as a •) corresponds to the empty sequence. Given a non-empty sequence we start in the origin and interpret the bits one by one: a 0 indicates a move to the left, and a 1 a move to the right. For example, the lower left corner of the structure shown above has address 0010101010.

Plotting all the itineraries (parity sequences) produced by Everett's function would fill the entire structure, at each level. The Collatz function, however, produces itineraries in which no two adjacent 1's are allowed:



2.4 *The Golden Mean Shift*

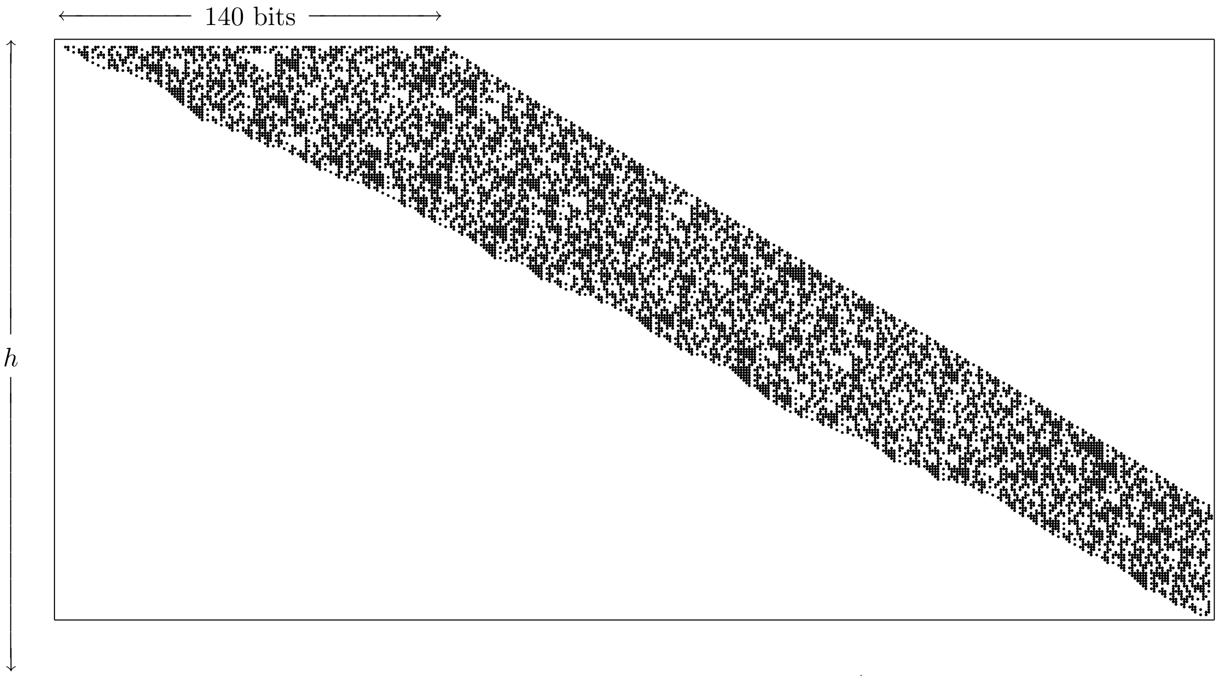
A *symbolic dynamical system* (X, σ) of the kind considered in this note consists of a set X of infinite sequences of symbols and a shift function σ that knocks off the first term of each sequence. As an example, let $\{0, 1\}$ be the symbol set, and let $X = \{0, 1\}^\infty$ be the set of all infinite 0-1 sequences of the form $x = c_0c_1c_2\ldots$, where $c_i = 0$ or 1. Define the shift function $\sigma : X \to X$ by $\sigma(x) = c_1c_2c_3\ldots$. Let G denote the subset of $\{0,1\}^\infty$ that consists of the sequences in which adjacent ones are forbidden. The set G together with the shift function σ defined above (G, σ) is called the *golden-mean shift* (see [4], [6]).

Define $B : \{0,1\}^\infty \to G$ by $B(z_1z_2z_3\ldots) = y_1y_2y_3\ldots$, where

$$y_n = \begin{cases} 10 & z_n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

It is easy to see that B is a bijection that translates trajectories generated by the function defined at (1) into trajectories generated by the Collatz function. Thus the two functions are in a sense *bisimilar* (that is, they essentially behave the same).

Here's a number bigger than 2^{140} during its trip through the Collatz iterator:

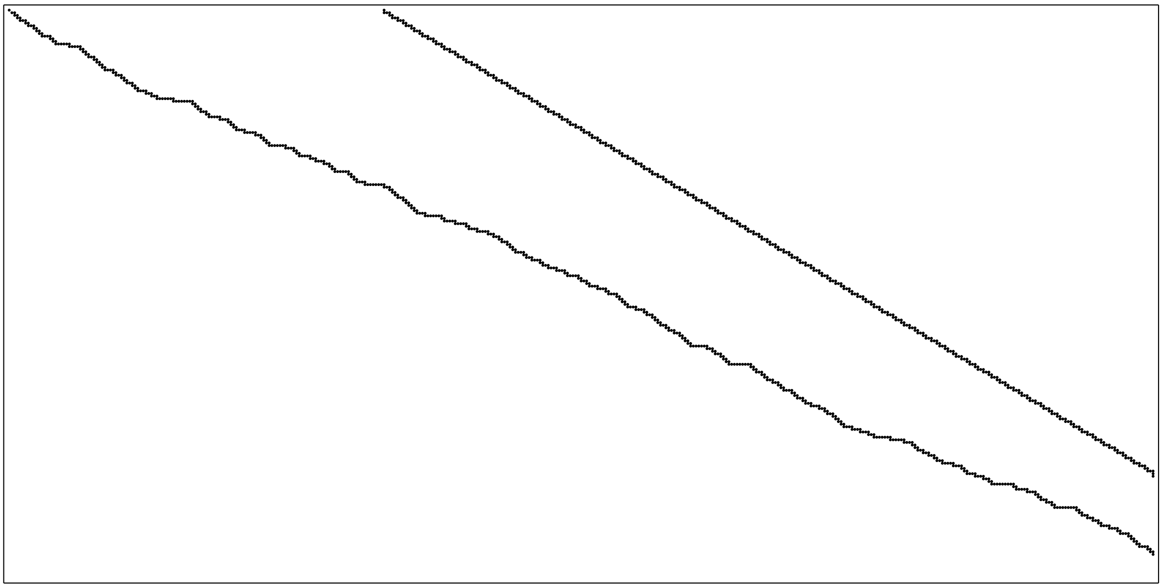


Relatively extensive tests using arbitrary precision arithmetic (of the kind provided by languages like *Chez Scheme*) indicate that the average behaviour of the diagram presented above could be summarized as follows:

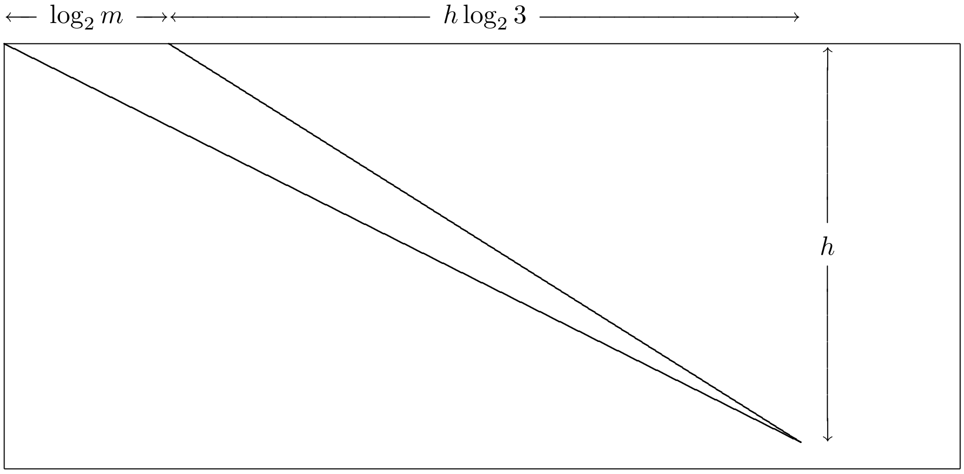
$$h \approx \frac{\log_2 m}{2 - \log_3 2}$$

The average behaviour, however, is not of great interest here. Still, in the equation above h denotes the number of times one multiplies by 3, starting with m, until the number reaches 1. What we are really interested in is the exact characterization of the *itineraries* for arbitrary integers m under the repeated application of the Collatz function. A precise determination of h appears to be difficult since itineraries (left side of the diagram above) seem to behave erratically, unlike the opposite edge of the diagram that has the average slope indicated a bit earlier.

Here's another diagram (drawn for another random number) in which only the lattice paths determined by the least and most significant digits of the number are shown. (The itinerary is the lattice path generated by the least significant digit.)



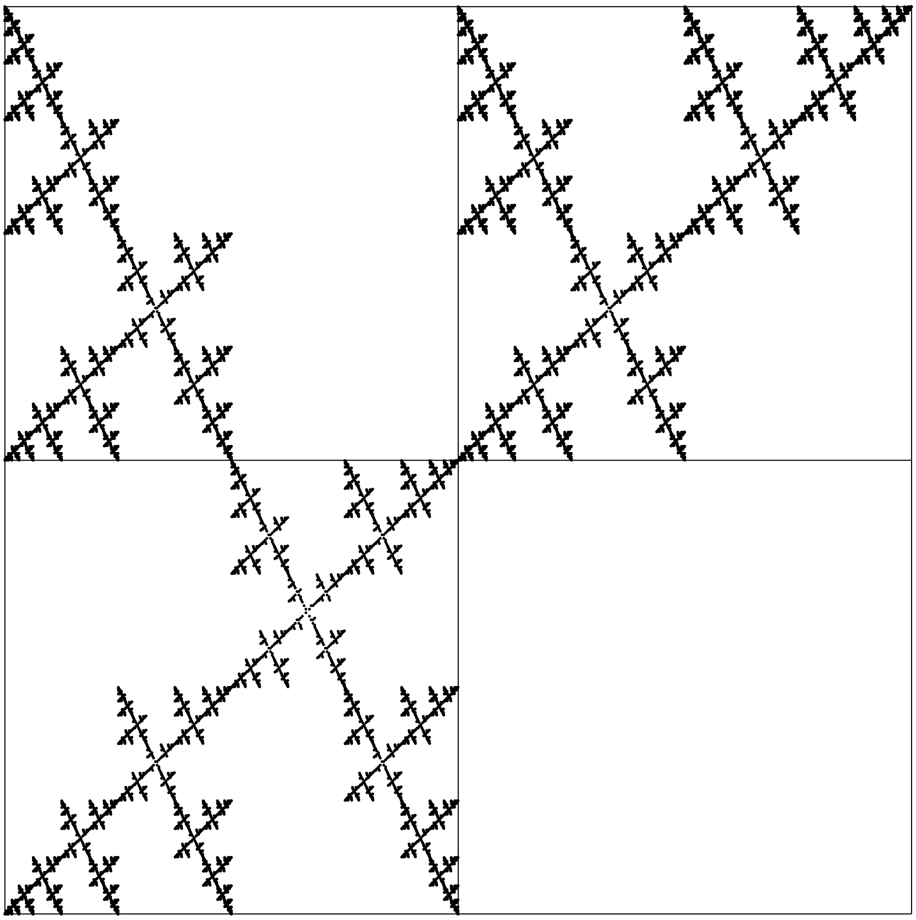
Although the itinerary seems to be a bit more tame this time around it still retains a distinctly unpredictable appearance. The average number of iterations conjectured above relies on the approximation of the slope of the itinerary for an average large number m (experimentally the slope is determined to be about 0.5).



But, as we indicated, this (average behaviour) aspect is entirely secondary. Recall that itineraries are essentially parity sequences (that is, sequences of binary digits). One can plot them on the fractal structure of section 2.3 if one identifies them with binary addresses in that structure—and that's what we do next.

2.5 Plotting the Golden Mean Shift

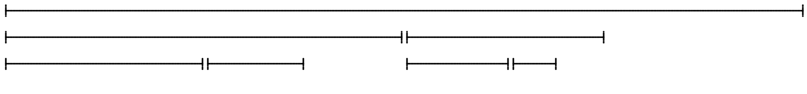
The set of itineraries generated by the Collatz IFS can be represented as follows:



The picture above is obtained by drawing only those points in the fractal structure of section 2.3 that correspond to admissible vectors (the set of itineraries produced by the Collatz function). As one can immediately see no points appear in the lower right quadrant (since all points in it would have addresses starting with 11). The structure is indeed self-similar and is the invariant set (attractor) of the following iterated function system (two dilations of ratios $\frac{1}{2}$ and $\frac{1}{4}$ respectively):

$$f_1(x) = \frac{x}{2} \qquad f_2(x) = \frac{x}{4} + \frac{1}{2} \tag{9}$$

Graphically we can represent the successive effect of the dilations as follows:



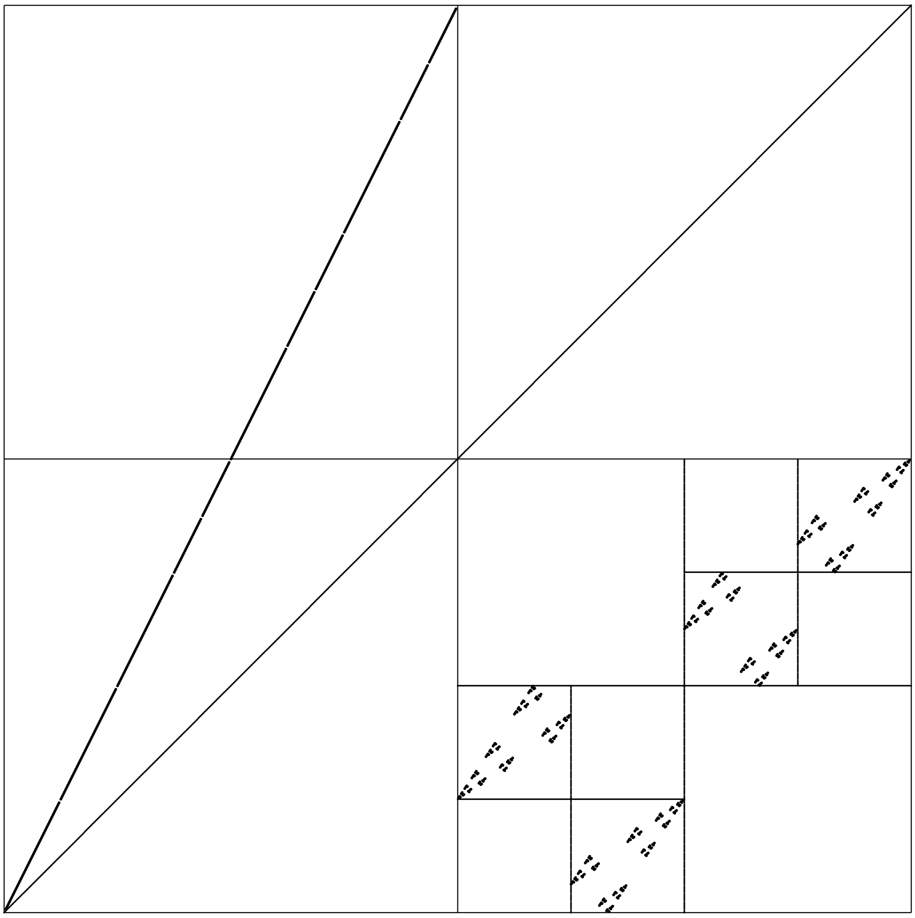
3 A Mirror Function On the Unit Interval

We now consider the function $\mu : [0, 1) \to [0, 1)$ defined as follows.

Let $x_0 x_1 \ldots x_u$ be the mirror image of the binary representation for a positive integer m and let $y_1 y_2 \ldots y_v$ represent the mirror image of the binary representation of $f(m)$, where f is the Collatz function as defined at (6). Obviously $u+1 = \lfloor \log_2 m \rfloor$ and likewise $v + 1 = \lfloor \log_2 f(m) \rfloor$.

Now let x be $0.x_0 x_1 \ldots x_u$ (obtained by adding a 0 (zero) and a decimal point in front of the mirror image of the binary representation of m) and let $y = 0.y_0 y_1 \ldots y_v$ obtained in the same way from the mirror image of the binary representation of $f(m)$. Rational numbers of the form $j/2^n$ are called *dyadic numbers*. It is clear that by construction both x and y are dyadic numbers in the interval $[0, 1)$.

We now make $\mu(x) = y$ for all x and y defined as above and plot it.



Now this graph is extremely important. It depicts μ over the interval $[0, 1)$. μ is a function that is discontinuous in every dyadic point of the interval $[\frac{1}{2}, 1)$ (since dyadic numbers admit two representations). On $I_0 = [0, \frac{1}{2})$ it is a straight line. On

$I_1 = [\frac{1}{2}, 1)$ it is self-similar, and for a simple reason: alternate sequences of zeros and ones $(01)^k$ turn into zeros through the Collatz function. For example $f(13) = 40$ so $\mu(0.1011) = 0.000101$. Notice that from here on the next three applications of μ amount to multiplication by $2^3 = 8$, then (since the pattern 101 is all by itself) μ takes us into 0.00001 and in four more steps we're in 0.1 (which is 0.5 in decimal).

The function μ has a 3-cycle $(\frac{1}{2}, \frac{1}{8}, \frac{1}{4})$ and careful investigation of the graph shown suggests the following: (a) that the only 3-cycle of μ consists of the three dyadic numbers $\frac{1}{2} \to \frac{1}{8} \to \frac{1}{4} \to \frac{1}{2}$, and (b) that all other dyadic numbers in $[0, 1)$ are eventually period-three points with their orbits ending with the 3-cycle above.

The function above is somewhat similar in its dynamics to the function

$$f(x) = \begin{cases} 2x & x \in I_0 = [0, \frac{1}{2}) \\ x - \frac{1}{2} & x \in I_1 = [\frac{1}{2}, 1) \end{cases} \tag{10}$$

so thinking of the dynamics of this function might help although it is not in any way a prerequisite for the understanding of the argument sketched below.

First notice that dividing I_1 in four yields four endpoints: $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$ and $\frac{7}{8}$. All four have orbits that end in the 3-cycle mentioned (this can be checked easily).

Suppose that $x \in [0, 1)$ is dyadic. Then $x = j/2^n$ in lowest terms, with $n \geq 2$, and $x = j/2^n$ is the midpoint of an interval of the form $(p/2^{n-1}, (p+1)/2^{n-1})$ that falls entirely in I_0 or I_1. Every such interval in I_1 has an exact counterpart in I_0 (for the function defined at (10) it's the interval shifted to the left by $\frac{1}{2}$) modulo the mixing properties of the function μ which really amount at each level to just two permutations: $(a, b, c, d) \to (b, a, c, d)$ and $(a, b, c, d) \to (a, b, d, c)$. Every visit of the dyadic interval $(p/2^{n-1}, (p + 1)/2^{n-1})$ to I_0 would double the interval and either keep it entirely in I_0 or send it entirely to I_1. Every visit to I_1 would simply shuffle the four endpoints of the interval as indicated. The dyadic number we started with, x, will continue to travel as one of such 4-endpoints of an increasingly bigger (and shuffled upon each visit to I_1) interval, until it is completely mapped onto I_1 itself, when x (now reaching one of the 4-endpoints of I_1) enters the 3-cycle indicated at the beginning. The trajectory of the ever growing interval can be easily traced on the balanced binary tree that maps finite binary sequences to the unit interval.

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